Higher order Josephson effects

Roman V Buniy[1,3] **and Thomas W Kephart**[2]

[1] Physics Department, Indiana University, Bloomington, IN 47405, USA
[2] Department of Physics and Astronomy, Vanderbilt University, Nashville, TN 37235, USA

E-mail: roman.buniy@gmail.com and tom.kephart@gmail.com

Abstract

Gaussian linking of superconducting loops containing Josephson junctions
with enclosed magnetic fields gives rise to interference shifts in the phase that
modulates the current carried through the loop, proportional to the magnitude
of the enclosed flux. We generalize these results to higher order linking of
a superconducting loop with several magnetic solenoids, and show that there
may be interference shifts proportional to the product of two or more fluxes.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Interference effects, both constructive and destructive, are mainstays for distinguishing
between quantum and classical phenomena. Examples include interfering scattering
amplitudes, both bosonic and fermionic, formation of condensates, entanglement, etc. The
Aharonov–Bohm effect [1] describes the self-interference of a charged particle that can travel
along two semiclassical paths whose combined path is Gaussian linked with a magnetic
solenoid carrying the flux Φ. The measurable phase shift is $\phi \propto \Phi$. We have argued in [2]
that there could exist generalizations to cases of higher order linkings. The simplest example
is a Borromean ring arrangement where the semiclassical path corresponds to one ring, which
has higher order linking with two flux tubes carrying fluxes Φ_1 and Φ_2, which make up the
other two rings. We found that the phase shift in this system is $\phi \propto \Phi_1\Phi_2$. Higher order
cases were explored in [3, 4] and shown to be related to commutator algebras of homotopy
generators of the configuration space $\mathbb{R}^3\backslash\{T_1 \cup T_2\}$, where T_1 and T_2 are the tubes containing
the fluxes. The same general logic can be applied to systems of superconductors, Josephson
junctions and magnetic fluxes where the Josephson effect can arise [5]. Here we will study

[3] Present address: School of Earth and Space Exploration, and Physics Department, Arizona State University, Tempe,
AZ 85287, USA.

interference in a symmetric arrangement of two identical semicircular superconductors joined by two identical Josephson junctions and derive the response of such systems. We conclude with a discussion of possible applications.

In the case of Gaussian linking of a loop of superconductor with a magnetic solenoid, the Mercereau effect [6] is due to the phase change in the macroscopic wavefunction, which is in turn related to the currents in the superconducting components. The effect is due to the presence of a vector potential A, which is the fundamental object responsible for the phase change. Exploration of higher order linking is again due to the presence of a vector potential but in these instances it requires careful choices of gauge.

2. The Josephson effect

It will be sufficient for our purposes to consider a macroscopic model of superconductors. Following Feynman [7], we approximate the superconductors coupled via a Josephson junction as a two-level system. Let ψ_1 and ψ_2 be the states, and E_1 and E_2 the energy levels of the superconductors. The Schrödinger equation for the coupled system of the two superconductors becomes

$$i\hbar(\partial\psi_1/\partial t) = E_1\psi_1 + K\exp(i\phi)\psi_2, \tag{1}$$

$$i\hbar(\partial\psi_2/\partial t) = E_2\psi_2 + K\exp(-i\phi)\psi_1, \tag{2}$$

where K is the coupling energy and ϕ is a phase, which arises from the most general Hermitian Hamiltonian 2×2 matrix. The dependence of ϕ on the vector potential A can be found from gauge invariance considerations. For a gauge transformation $A \mapsto A + \nabla f$, $\psi_1 \mapsto \psi_1\exp(iqf/2\hbar c)$, $\psi_2 \mapsto \psi_2\exp(-iqf/2\hbar c)$ with an arbitrary function f of space coordinates, we find $\phi \mapsto \phi + qf/\hbar c$, from which it follows that $\phi = (q/\hbar c)\int A \cdot dx$. Here $q = 2e$ is the charge of an electron pair.

After the substitutions $\psi_1 = |\psi_1|\exp(i\theta_1)$ and $\psi_2 = |\psi_2|\exp(i\theta_2)$, the Schrödinger equation becomes

$$\hbar(\partial|\psi_1|^2/\partial t) = 2K|\psi_1||\psi_2|\sin\theta, \tag{3}$$

$$\hbar(\partial|\psi_2|^2/\partial t) = -2K|\psi_1||\psi_2|\sin\theta, \tag{4}$$

$$-\hbar|\psi_1|(\partial\theta_1/\partial t) = E_1|\psi_1| + K|\psi_2|\cos\theta, \tag{5}$$

$$-\hbar|\psi_2|(\partial\theta_2/\partial t) = E_2|\psi_2| + K|\psi_1|\cos\theta, \tag{6}$$

where $\theta = \phi + \theta_2 - \theta_1$. The current from superconductor 1 to superconductor 2, which is equal to minus the current from superconductor 2 to superconductor 1, is thus $I = (2K/\hbar)|\psi_1||\psi_2|\sin\theta$. (In a self-consistent computation, a current from a battery which connects the two superconductors is also included. The result for the superconducting current is precisely I; see, for example, [8].) The electron densities in the two superconductors are approximately equal and independent of time; let ρ be this common constant. This gives $I = I_0\sin\theta$, where $I_0 = 2K\rho/\hbar$. Integrating the phase equations, we find

$$\theta(t) = \phi + \theta(0) + \hbar^{-1}\int_0^t dt'(E_1(t') - E_2(t')). \tag{7}$$

The quantity $(E_1(t) - E_2(t))/q$ represents an electric potential applied to the junction. The dc and ac Josephson effects [5] arise for $|E_1(t) - E_2(t)| \ll K$ and $|E_1(t) - E_2(t)| \gg K$, respectively.
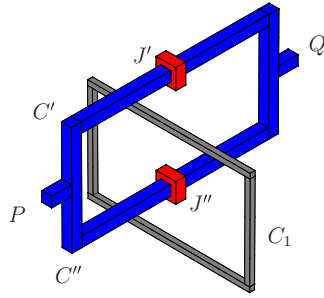
Figure 1. A diagram of an experimental setup for the detection of the Josephson effect. C' and C'' are the paths from the point P to the point Q through the superconductors with the Josephson junctions J' and J'' and the total current I from P to Q. C_1 is the magnetic solenoid carrying the flux Φ_1. The Josephson effect (for a review see [9]) is due to the first-order (Gaussian) linking of the closed curves $C = C'C''^{-1}$ and C_1.

Our interest here is in the Josephson effect with zero potential across the junction, $E_1(t) - E_2(t) = 0$, and nonzero magnetic field constrained to the opening of the superconducting ring with two Josephson junctions, see figure 1. Let θ' and θ'' be the phase changes due to the vector potential A_1 of the currents through the junctions J' and J''. The phase changes from the point P to the point Q along the paths C' and C'' are

$$\phi' = \theta' + (q/\hbar c) \int_{C'} A_1 \cdot dx, \tag{8}$$

$$\phi'' = \theta'' + (q/\hbar c) \int_{C''} A_1 \cdot dx. \tag{9}$$

Since the wavefunction is single valued, this requires $\phi' = \phi''$, and so we find that $\theta'' - \theta' = 2\pi \Phi_1/\Phi_0$. Here $\Phi_1 = \oint_C A_1 \cdot dx$ is the flux due to the solenoid along C_1 passing through a surface spanned by a closed curve $C = C'C''^{-1}$ and $\Phi_0 = 2\pi\hbar c/q$ is the flux quantum. The total current from the point P to the point Q is

$$I = I_0 \sin\left(\tfrac{1}{2}(\theta' + \theta'')\right) \cos\left(\pi \Phi_1/\Phi_0\right). \tag{10}$$

For a fixed value of Φ_1, the corresponding maximal total current is

$$I_{\text{max}} = I_0 |\cos\left(\pi \Phi_1/\Phi_0\right)|, \tag{11}$$

which itself has maxima when $\Phi_1 = n\Phi_0, n \in \mathbb{Z}$.

The flux is actually $\Phi_1 = \Phi_{1,\text{ext}} + LI$ where $\Phi_{1,\text{ext}}$ is the external flux through the loop, L is the self-inductance, but here and in what follows we assume that L is negligible. (We have made a number of simplifying assumptions, for example, that self-inductance of SQUID components are negligible, none of which, if relaxed, affect our basic conclusions.)

We will call the phenomena reviewed in this section the first-order Josephson effects to distinguish them from their generalizations which we now proceed to describe.

3. The second and higher order Josephson effects

Now consider the case where we have two solenoids carrying the magnetic fluxes Φ_1 and Φ_2 and whose center lines run along C_1 and C_2, and a superconducting ring along the closed curve $C = C'C''^{-1}$ with two Josephson junctions J' and J'' in parallel as shown
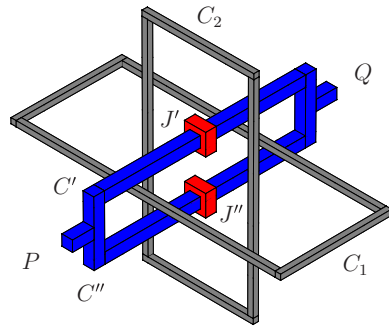
Figure 2. A diagram of an experimental setup for the detection of the second-order Josephson effect. C_1 and C_2 are the magnetic solenoids. C' and C'' are the paths from the point P to the point Q through the superconductors connected by Josephson junctions J' and J''. The total current from P to Q is I. C_1 and C_2 are the magnetic solenoids carrying the fluxes Φ_1 and Φ_2. The second-order Josephson effect is due to the second-order linking of the set of three closed curves $C = C'C''^{-1}$, C_1 and C_2.
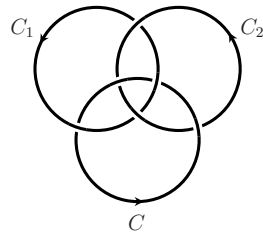

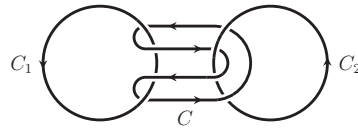**Figure 3.** Classic form of the Borromean rings.

Figure 4. Borromean rings altered by fixing C_1 and C_2 as hard rings, allowing C to be flexible and then pulling on C_1 and C_2.

in figure 2. The two solenoids and the superconducting ring are in a Borromean ring [10] configuration. Note that in this arrangement neither C_1 nor C_2 has Gaussian linking with the superconducting ring C, nor do C_1 and C_2 link with each other. However, the set of three rings C, C_1, C_2 is indeed linked. This second-order linking and its higher order generalizations are what will lead to our results. We will find that even though our system lacks first-order (Gaussian) linking, a phase difference can still be nonzero upon traveling around the superconductor.

To begin the analysis of the Borromean rings, we first need to show the precise form of the path C. In particular, we will show that C has no Gaussian linking with C_1 or C_2, yet the three paths are still inextricably linked. First, we draw the three rings, see figure 3. Now assume that C is flexible, but C_1 and C_2 are not, and pull on C_1 and C_2 to arrive at the configuration shown in figure 4. Now pinch C as shown in figure 5 and note that we have labeled C as a set of four components. Following the full circuit around C, we first travel along a_1, followed by
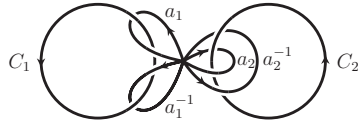
Figure 5. Borromean rings where path C has become a comutator $a_1 a_2 a_1^{-1} a_2^{-1}$.

a_2, followed by a_1^{-1} and finally by a_2^{-1}. (Construction with string and wire can be useful at this point.) Note that a_1 links with C_1 and a_2 links with C_2 in the positive sense, respectively, while a_1^{-1} and a_2^{-1} link with C_1 and C_2 in the negative sense, respectively. As a result, the entire path C runs through C_1 once in the positive and once in the negative sense, so the total Gaussian linking of C with C_1 vanishes. Likewise, the Gaussian linking of C with C_2 vanishes. Nevertheless, the total path C is not trivial. This is because the paths a_1 and a_2 do not commute [11]. In fact, we have transformed the path C into a commutator path, which we can write in a multiplicative form as

$$C = a_1 a_2 a_1^{-1} a_2^{-1}. \tag{12}$$

We now must find a way to detect the phase due to the higher order linking of this commutator path with the two solenoids. To find this phase, we first need to choose a gauge that is capable of detecting it. As we will see, such a gauge is [2]

$$\boldsymbol{A}_{12} = \tfrac{1}{2} k_2 (\gamma_1 \boldsymbol{A}_2 - \gamma_2 \boldsymbol{A}_1). \tag{13}$$

Here $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the vector potentials due to the solenoids along C_1 and C_2, and the quantities γ_1 and γ_2 (not to be confused with integrals (8) and (9)) are defined by

$$\gamma_j = \delta_j + (q/\hbar c) \int_\Gamma \boldsymbol{A}_j \cdot \mathrm{d}\boldsymbol{x}, \tag{14}$$

where δ_1 and δ_2 are constants, and Γ is a path that runs along C. The quantity k_2 is a normalization constant, the value of which we discuss below. Our task is then to integrate $\boldsymbol{A}_{12}$ around C, written in terms of a commutator path, and show that it delivers a non-vanishing phase

$$\phi_{12} = \theta' - \theta'' = (q/\hbar c) \oint_C \boldsymbol{A}_{12} \cdot \mathrm{d}\boldsymbol{x}, \tag{15}$$

which only depends on the topology of the path.

We begin at the initial position $(\gamma_1, \gamma_2) = (\delta_1, \delta_2)$. We first integrate along a_1 to get $(\delta_1 + \Phi_1, \delta_2)$, picking up an area $\delta_2 \Phi_1$, which corresponds to a contribution of $-\tfrac{1}{2}\delta_2 \Phi_1$ to the phase $k_2^{-1}\phi_{12}$. Next, a_2 takes us to $(\delta_1 + \Phi_1, \delta_2 + \Phi_2)$ and it generates a contribution $\tfrac{1}{2}(\delta_1 + \Phi_1)\Phi_2$. Next, a_1^{-1} takes us to $(\delta_1, \delta_2 + \Phi_2)$ and contributes $\tfrac{1}{2}\Phi_1(\delta_2 + \Phi_2)$. Finally, a_2^{-1} returns us to (δ_1, δ_2) and contributes $-\tfrac{1}{2}\delta_1 \Phi_2$. The total contribution to the phase is $k_2^{-1}\phi_{12} = \Phi_1 \Phi_2$ for traversing the full loop C. We find that

$$\phi_{12}(C) = k_2 \frac{e^2}{\hbar^2 c^2} \Phi_1 \Phi_2 \tag{16}$$

once physical units have been restored. Note that this result is independent of the initial point (δ_1, δ_2) where we choose to start the path C. It is straightforward to check that an integral of $\boldsymbol{A}_{12}$ over a closed path is path independent if and only if the path can be written as a commutator. This is precisely the case for the Boromean rings, and hence $\boldsymbol{A}_{12}$ is an appropriately chosen gauge potential.

Now returning to our specific generalized Josephson configuration in the form of Borromean rings, we need only replace A_1 in the usual Josephson configuration with A_{12} in the phase integrals (8) and (9) so that $\theta'' - \theta'$ becomes $4\pi^2 k_2 \Phi_1 \Phi_2 / \Phi_0^2$. Hence, the total current from the point P to the point Q is

$$I = I_0 \sin\left(\tfrac{1}{2}(\theta' + \theta'')\right) \cos\left(2\pi^2 k_2 \Phi_1 \Phi_2 / \Phi_0^2\right). \tag{17}$$

For fixed values of Φ_1 and Φ_2, the maximal total current flowing in the superconductor is

$$I_{\max} = I_0 \left| \cos\left(2\pi^2 k_2 \Phi_1 \Phi_2 / \Phi_0^2\right) \right|. \tag{18}$$

The smallest value of the constant $k_2 > 0$ for which the fluxes $\Phi_1 = m_1 \Phi_0$, $\Phi_2 = m_2 \Phi_0$, where $m_1, m_2 \in \mathbb{Z}$, lead to maxima of the quantity $I_{\max}$ is $k_2 = (2\pi)^{-1}$. This is precisely the value we obtained in [2] by imposing an analog of the Dirac string condition on the second-order phase for the Aharonov–Bohm effect. Nevertheless, the value of k_2 must ultimately be determined by experiment.

Also, for the value $k_2 = (2\pi)^{-1}$, if either Φ_1 or Φ_2 is equal to Φ_0 or $-\Phi_0$, then in terms of the other flux, appropriately relabeled, expression (18) for the second-order $I_{\max}$ reduces to expression (11) for the first-order $I_{\max}$.

An essential feature of the above result is that the current I is a periodic function of the quantity $\pi k_2 \Phi_1 \Phi_2 / \Phi_0^2$ with period 1. We can also derive this property by modifying the method used by Block [12] for the first-order Josephson effect as follows.

The total gauge potential includes internal and external parts, $A = A_{\mathrm{in}} + A_{\mathrm{ext}}$, the external magnetic field being due to the external sources. Assuming that the external field $\nabla \times A_{\mathrm{ext}}$ vanishes inside the superconductors, we can write $A_{\mathrm{ext}} = \nabla \gamma_{\mathrm{ext}}$. As a result, A is a gauge transformation of A_{in}, and so

$$\psi(A_{\mathrm{in}} + A_{\mathrm{ext}}) = \psi(A_{\mathrm{in}}) \exp(\mathrm{i}q \gamma_{\mathrm{ext}} / \hbar c). \tag{19}$$

Since $\psi(A)$ is single valued, we find that $\psi(A_{\mathrm{in}})$ is multiplied by the factor $\exp\left(-4\mathrm{i}\pi^2 k_2 \Phi_1 \Phi_2 / \Phi_0^2\right)$ after the charge q travels around a closed curve C. This factor is a periodic function of $\pi k_2 \Phi_1 \Phi_2 / \Phi_0^2$ with period 2. This implies the same periodicity property for the wavefunction $\psi(A_{\mathrm{in}})$ and the energy E. Assuming time reversal symmetry as in [12], we find that the free energy and thus the current, which is given by minus the derivative of the free energy with respect to the external flux, are both periodic functions of $\pi k_2 \Phi_1 \Phi_2 / \Phi_0^2$ with period 1, in agreement with the result proved earlier.

More generally, it is straightforward to arrange n solenoids with the fluxes $\Phi_1, \ldots, \Phi_n$ and a superconducting ring in such a way that they are linked with nonzero nth-order linking [10]. We similarly find the phase difference

$$\theta'' - \theta' = (2\pi)^n k_n \Phi_1 \cdots \Phi_n / \Phi_0^n, \tag{20}$$

the current

$$I = I_0 \sin\left(\tfrac{1}{2}(\theta' + \theta'')\right) \cos\left(\tfrac{1}{2}(2\pi)^n k_n \Phi_1 \cdots \Phi_n / \Phi_0^n\right) \tag{21}$$

and its maximal value for fixed values of $\Phi_1, \ldots, \Phi_n$

$$I_{\max} = I_0 \left| \cos\left(\tfrac{1}{2}(2\pi)^n k_n \Phi_1 \cdots \Phi_n / \Phi_0^n\right) \right|. \tag{22}$$

Other properties of these systems can be investigated.

The smallest value of the constant $k_n > 0$ for which the fluxes $\Phi_j = m_j \Phi_0$, where $m_j \in \mathbb{Z}$, lead to the maxima of the quantity $I_{\max}$ is $k_n = (2\pi)^{1-n}$. Again this is precisely the value we obtained in [2, 3] by imposing an analog of the Dirac string condition on the phase for the nth-order Aharonov–Bohm effect. Nevertheless, as pointed out with the k_2 case above, the value of k_n must ultimately be determined by experiment.

Also, for the value $k_n = (2\pi)^{1-n}$, if one of the fluxes is equal to Φ_0 or $-\Phi_0$, then in terms of the remaining fluxes, appropriately relabeled, expression (22) for the nth-order I_{max} reduces to the analogous expression (22) for the $(n-1)$st-order I_{max}.

Similar to the case $n = 2$ above, we can modify the method used by Block and prove that for any n the current I is a periodic function of the quantity $\frac{1}{2}(2\pi)^{n-1}k_n\Phi_1\cdots\Phi_n/\Phi_0^n$ with period 1.

4. Discussion and conclusion

Our previous work on the subject of higher order phases [2, 3] focused on generalizations of the Aharonov–Bohm experiment because we believe higher order interference effects are most easily understood in the language of the Aharonov–Bohm effect where wavefunction self-interference is most evident as measured by electron holography. In contrast, this paper, while being intimately related to the previous work, is focused on higher order phase effects in superconducting quantum interference experiments. Specifically, we have generalized the Josephson effect from the first-order (Gaussian) linking case to second-order linking of the Borromean ring type and have shown that the second-order phase in the current depends on the product of the magnetic fluxes in the two solenoids as shown in figure 2. We also discussed higher order analogs where a superconducting loop containing Josephson junctions links with several magnetic solenoids and where the resulting interference shifts are proportional to the product of multiple fluxes. We hope that the present paper is useful to the low-temperature community since it translates and extends our previous results into an alternate form that suggests how one would measure higher order interference effects via currents flowing through SQUIDs. The above results should be useful for providing a broader understanding of higher order interference effects.

The Josephson junction experimental setup we suggest for measuring higher order phases is perhaps somewhat over simplified but still sufficiently realistic that it could be used as a starting point for an experimentalist who wants to make a practically designed experiment that incorporates contingencies to deal with potential issues, such as stray fields modifying the Hamiltonian, experimental separation of first and higher order effects, etc. We should mention that if higher order interference effects are found in one system, then it would suggest that there could be many other systems with similar behavior, from neutron beams to atomic Bose–Einstein condensates.

One can conceive of a number of applications for devices built to take advantage of higher order linking. Such a system could be less invasive than first-order devices because it could keep the SQUID some distance from an experimental sample. Possible applications include both rf and dc SQUIDs that measure higher order linking of multiple fluxes. Under some circumstances such devices could be useful in measurements of complex biological systems, or any systems where direct Gaussian linking of a magnetic flux with a SQUID is impractical, but where higher order linking is possible. For example, one could have a system of (i) a fixed but adjustable flux tube, i.e. a solenoid; (ii) an unknown flux to be measured; and (iii) a SQUID. If the three components can be arranged to have higher order linking, then the unknown flux could be measured, even though it has no Gaussian linking with the SQUID.

Acknowledgments

References

[1] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
[2] Buniy R V and Kephart T W 2008 *Phys. Lett.* A **372** 2583 (arXiv:hep-th/0611334)
[3] Buniy R V and Kephart T W 2008 *Phys. Lett.* A **372** 4775 (arXiv:hep-th/0611335)
[4] Buniy R V and Kephart T W 2006 arXiv:hep-th/0611336
[5] Josephson B D 1962 *Phys. Lett.* **1** 251
[6] Jaklevic R C, Lambe J, Silver A H and Mercereau J E 1964 *Phys. Rev. Lett.* **12** 159
     Jaklevic R C, Lambe J, Silver A H and Mercereau J E 1964 *Phys. Rev. Lett.* **12** 274
[7] Feynman R P, Leighton R B and Sands M 1965 *Feynman Lectures on Physics* vol 3 chapter 21 (Reading, MA: Addison-Wesley)
[8] Ohta H A 1976 *IC-SQUID* **76** 35
[9] Golubov A A, Kupriyanov M Yu and Il'ichev E 2004 *Rev. Mod. Phys.* **76** 411
[10] Rolfson D 1976 *Knots and Links* (Wilmington, DE: Publish or Perish)
[11] Hatcher A 2002 *Algebraic Topology* (Cambridge: Cambridge University Press)
[12] Block F 1970 *Phys. Rev.* B **2** 109